

Distance spectra and Distance energy of Integral Circulant Graphs

ALEKSANDAR ILIĆ [‡]

*Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia
e-mail: aleksandari@gmail.com*

January 19, 2013

Abstract

The distance energy of a graph G is a recently developed energy-type invariant, defined as the sum of absolute values of the eigenvalues of the distance matrix of G . There was a vast research for the pairs and families of non-cospectral graphs having equal distance energy, and most of these constructions were based on the join of graphs. A graph is called circulant if it is Cayley graph on the circulant group, i.e. its adjacency matrix is circulant. A graph is called integral if all eigenvalues of its adjacency matrix are integers. Integral circulant graphs play an important role in modeling quantum spin networks supporting the perfect state transfer. In this paper, we characterize the distance spectra of integral circulant graphs and prove that these graphs have integral eigenvalues of distance matrix D . Furthermore, we calculate the distance spectra and distance energy of unitary Cayley graphs. In conclusion, we present two families of pairs (G_1, G_2) of integral circulant graphs with equal distance energy – in the first family G_1 is subgraph of G_2 , while in the second family the diameter of both graphs is three.

Keywords: distance matrix; distance energy; unitary Cayley graph; integral circulant graph.

AMS subject classification: 05C50, 05C12.

1 Introduction

Let G be a simple undirected graph with n vertices. The vertices of G are labeled as $0, 1, \dots, n-1$. The distance between the vertices i and j is the length of a shortest path between them, and is denoted by $d(i, j)$. The diameter of G , denoted by $\text{diam}(G)$, is the maximum distance between any pair of vertices of G .

Let A be the adjacency matrix of G , and $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of the graph G . The energy of G is defined as the sum of absolute values of its eigenvalues [1],

$$E(G) = \sum_{i=1}^n |\lambda_i|.$$

The energy is a graph parameter stemming from the Hückel molecular orbital approximation for the total π -electron energy (for recent survey on molecular graph energy see [2] and [3]).

The distance matrix of a graph G is the square matrix $D(G) = [d(i, j)]_{i,j=1}^n$. The eigenvalues of the distance matrix $D(G)$, labeled as $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$, are said to be the distance or D -eigenvalues of G and to form the distance or D -spectrum of G [4]. The sum of distance eigenvalues is zero, $\sum_{i=1}^n \mu_i = 0$. The characteristic polynomial and the eigenvalues of the distance matrix of a graph were considered in [5]–[11].

Definition 1.1 *The distance energy $DE(G)$ of a graph G is the sum of absolute values of the eigenvalues of the distance matrix of G .*

Distance energy is a useful molecular descriptor in QSPR modeling, as demonstrated by Consonni and Todeschini in [12]. To avoid trivial cases, we say that the graphs G and H of the same order are D-equienergetic if $DE(G) = DE(H)$, while they have distinct spectra of distance matrices.

Our motivation for this research came from various constructions of non-cospectral, equienergetic graphs with equal number of vertices [13]–[16]. Indulal et al. [17] constructed pairs of D-equienergetic graphs on n vertices for $n \equiv 1 \pmod{3}$ and for $n \equiv 0 \pmod{6}$. Ramane et al. [18] proved that if G_1 and G_2 are r -regular graphs on n vertices and $\text{diam}(G_i) \leq 2$, $i = 1, 2$, then $DE(L^k(G_1)) = DE(L^k(G_2))$ for $k \geq 1$, where $L^k(G)$ is the k -th iterated line graph of G . In [19] the authors obtain the eigenvalues of the distance matrix of the join of two graphs whose diameter is less than or equal to two, and construct pairs of non D-cospectral, D-equienergetic graphs on n vertices for all $n \geq 9$. Stevanović and Indulal [20] further generalized this result and described the distance spectrum and energy of the join-based compositions of regular graphs in terms of their adjacency spectrum. Those results are used to show that there exist a number of families of sets of non-cospectral graphs with equal distance energy, such that for any $n \in N$, each family contains a set with at least n graphs. All these constructions from the literature are based on graph products and most of presented graphs have diameter two.

A graph is called *circulant* if it is Cayley graph on the circulant group, i.e. its adjacency matrix is circulant. A graph is called *integral* if all eigenvalues of its adjacency matrix are integers. Integral graphs are extensively studied in the literature and there was a vast research for specific classes of graphs with integral spectrum [21].

Integral circulant graphs were imposed as potential candidates for modeling quantum spin networks with periodic dynamics. For the certain quantum spin system, the necessary condition for the existence of perfect state transfer in qubit networks is the periodicity of the system dynamics (see [22]). Relevant results on this topic were given in [23], where it was shown that a quantum network topology based on the regular graph with at least four distinct eigenvalues is periodic if and only if it is integral. Various properties of integral circulant graphs were investigated in [24]–[28].

Integral circulant graphs arise as a generalization of unitary Cayley graphs, recently studied by Klotz and Sander [25]. Let D be a set of positive, proper divisors of the integer $n > 1$. Define the graph $ICG_n(D)$ to have vertex set $Z_n = \{0, 1, \dots, n - 1\}$ and edge set

$$E(ICG_n(D)) = \{\{a, b\} \mid a, b \in Z_n, \gcd(a - b, n) \in D\}.$$

In this paper our intention is to move a step towards in the investigation of the graph theoretical properties of integral circulant graphs that are important parameters of quantum networks. Namely, we deal with the distance matrix of integral circulant graph $ICG_n(D)$ and distance spectra. The integral circulant graph $ICG_{10}(1)$ is shown on Figure 1, together with its distance matrix.

The paper is organized as follows. In Section 2 we present some preliminary results on integral circulant graphs, while in Section 3 we prove that distance matrix of ICG graphs have integral spectra, and characterize the vertices at distance k from the starting vertex 0. Our main result is the description of the distance spectrum of $ICG_n(D)$. In Section 4 we calculate the distance energy, the distance spectral radius and the Wiener index of unitary Cayley graphs. Note that the distance energy of $ICG_n(1)$, where n is even with odd prime divisor, is not fully resolved since the sign of expression $2(\varphi(n) - 1) - \frac{n}{2}$ can vary. Finally, in Section 5 we present two families of non-cospectral distance equienergetic graphs ($ICG_{3p}(1), ICG_{3p}(1, p)$) and ($ICG_{2pq}(1, p), ICG_{2pq}(1, q)$) for arbitrary primes $p > q > 3$. These constructions are not based

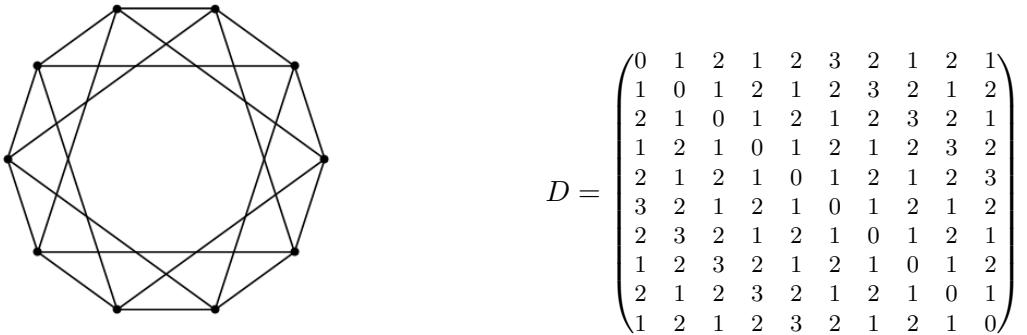


Figure 1: The integral circulant graph $ICG_{10}(1)$ and its distance matrix.

on the graph products, and in the first pair we have that $ICG_{3p}(1)$ is subgraph of $ICG_{3p}(1, p)$, while in the second pair the diameter of both graphs is three.

2 Preliminaries

Let us recall that for a positive integer n and subset $S \subseteq \{0, 1, 2, \dots, n-1\}$, the circulant graph $G(n, S)$ is the graph with n vertices, labeled with integers modulo n , such that each vertex i is adjacent to $|S|$ other vertices $\{i + s \pmod{n} \mid s \in S\}$. The set S is called a symbol of $G(n, S)$.

So [24] has characterized integral circulant graphs. Let

$$G_n(d) = \{k \mid gcd(k, n) = d, 1 \leq k < n\}$$

be the set of all positive integers less than n having the same greatest common divisor d with n . Let D_n be the set of positive divisors d of n , with $d \leq \frac{n}{2}$.

Theorem 2.1 A circulant graph $G(n, S)$ is integral if and only if

$$S = \bigcup_{d \in D} G_n(d)$$

for some set of divisors $D \subseteq D_n$.

It follows that the degree of $ICG_n(D)$ is equal to $\sum_{d \in D} \varphi(n/d)$, where $\varphi(n)$ denotes the Euler phi function. Authors in [28] proved that an integral circulant graph $ICG_n(D) = ICG_n(d_1, d_2, \dots, d_k)$ is connected if and only if $gcd(d_1, d_2, \dots, d_k) = 1$.

Let D be an arbitrary set of divisors $\{d_1, d_2, \dots, d_k\}$ of n . We establish the distance matrix $D_n(D)$ of integral circulant graph $ICG_n(D)$ with respect to the natural order of the vertices $0, 1, \dots, n-1$. The entries a_0, a_1, \dots, a_{n-1} of the first row of $D_n(D)$ generate the entries of the other rows by a cyclic shift.

$$D_n(D) = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1} & a_0 & \dots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & a_0 \end{pmatrix}$$

For more details on circulant matrices see [29]. There is an explicit formula for the eigenvalues μ_r , $0 \leq r \leq n-1$, of a circulant matrix such as D_n . Define the polynomial $P_n(z)$ by the entries of the first row of D_n ,

$$P_n(z) = \sum_{j=0}^{n-1} a_j \cdot z^j$$

The eigenvalues of D_n are given by

$$\mu_r = P_n(\omega^r) = \sum_{j=0}^{n-1} a_j \cdot \omega^{rj}, \quad 0 \leq r \leq n-1. \quad (1)$$

Ramanujan's sum, usually denoted $c(r, n)$, is a function of two positive integer variables r and n defined by the formula

$$c(r, n) = \sum_{\substack{a=1 \\ \gcd(a, n)=1}}^n e^{\frac{2\pi i}{n} \cdot ar} = \sum_{\substack{a=1 \\ \gcd(a, n)=1}}^n \omega_n^{ar},$$

where ω_n denotes a complex primitive n -th root of unity. These sums take only integral values,

$$c(r, n) = \mu\left(\frac{n}{\gcd(r, n)}\right) \cdot \frac{\varphi(n)}{\varphi\left(\frac{n}{\gcd(r, n)}\right)},$$

where μ denotes the Möbius function. In the next section, we will use the well-known summation [30]

$$s(r, n) = \sum_{i=0}^{n-1} \omega_n^{ir} = \begin{cases} 0 & \text{if } r \nmid n \\ n & \text{if } r \mid n \end{cases}$$

In [25] it was proven that *gcd-graphs* (the same term as integral circulant graphs $ICG_n(D)$) have integral spectrum,

$$\lambda_k = \sum_{d \in D} c\left(k, \frac{n}{d}\right), \quad 0 \leq k \leq n-1. \quad (2)$$

3 Distance spectra of ICG

Let $N_p(i)$ be the set of vertices that are at distance p from the starting vertex i . We will prove that

$$N_p(0) = \bigcup_{i=1}^{s_p} G_n(d_i^{(p)}),$$

for some divisor set of n

$$D^{(p)} = \{d_1^{(p)}, d_2^{(p)}, \dots, d_{s_p}^{(p)}\}.$$

For $p = 1$, this fits the definition of integral circulant graph $ICG_n(D)$ for divisor set $D^{(1)} = D$. Let a be an arbitrary vertex at the distance p from the starting vertex 0. Assume that $\gcd(a, n) = d$. It follows that there exist vertices $0 \equiv v_0, v_1, \dots, v_{p-1}, v_p \equiv a$, such that for all $0 \leq r \leq p-1$, the vertices v_p and v_{p+1} are adjacent. In other words,

$$\gcd(v_r - v_{r+1}, n) = d_{i_r}, \quad \text{for } r = 0, 1, \dots, p-1, \quad 1 \leq i_r \leq k.$$

Let b be an arbitrary vertex (different than a) such that $\gcd(b, n) = d$. This means that there are integers a' and b' relatively prime with n , such that $a = a' \cdot d$ and $b = b' \cdot d$. Since $\gcd(a', n) = 1$, from Bézout's identity we have

$$a' \cdot a^* \equiv 1 \pmod{n},$$

where a^* is the inverse of a' modulo n . Now, consider the following vertices $0 \equiv u_0, u_1, \dots, u_{p-1}, u_p \equiv b$, defined as

$$u_r = v_r \cdot a^* b'.$$

For $r = 0$ we have $u_0 = 0$, and for $r = p$ we have that $u_p = a \cdot a^* b' = d \cdot (a' \cdot a^*) \cdot b' = d \cdot b' = b$.
 For $r = 0, 1, \dots, p-1$ holds

$$\gcd(u_r - u_{r+1}, n) = \gcd((v_r - v_{r+1}) \cdot a^* b', n) = \gcd(v_r - v_{r+1}, n) = d_{i_r}.$$

This proves that the distance from 0 to the vertex b is less than or equal to k , $d(0, b) \leq d(0, a)$.
 Similarly, we prove that $d(0, a) \leq d(0, b)$ and finally, we get $d(0, a) = d(0, b) = k$.

It is easy to see that the set of all divisors is the union of all sets $D^{(p)}$, $p = 1, 2, \dots, \text{diam}(G)$.
 Using the relation (1), we get the distance spectra of $ICG_n(D)$:

$$\mu_r = 1 \cdot \sum_{i=1}^{s_1} c\left(r, \frac{n}{d_i^{(1)}}\right) + 2 \cdot \sum_{i=1}^{s_2} c\left(r, \frac{n}{d_i^{(2)}}\right) + \dots + \text{diam}(G) \cdot \sum_{i=1}^{s_{\text{diam}(G)}} c\left(r, \frac{n}{d_i^{(\text{diam}(G))}}\right).$$

This proves the main result of this section.

Theorem 3.1 *Integral circulant graph $ICG_n(D)$, where D is an arbitrary set of divisors of n , has integral distance spectra.*

The Wiener index of G is the sum of distances between all pairs of vertices,

$$W(G) = \sum_{a,b \in V} d(a, b).$$

The Wiener index is considered as one of the most used topological indices with high correlation with many physical and chemical properties of molecular compounds (for recent results and applications of Wiener index see [31]).

In [10] it is proven that $\mu_0 \geq \frac{2W(G)}{n}$, with equality if and only if all row sums are equal.
 Since the distance matrix of $ICG_n(D)$ is also a circulant matrix, it follows that

$$W(G) = \frac{n \cdot \mu_0}{2}.$$

4 Distance energy of UCG

The Unitary Cayley graph $ICG_n(1) \equiv X_n$ has the vertex set $V(X_n) = Z_n$ and the edge set

$$E(X_n) = \{(a, b) : a, b \in Z_n, \gcd(a - b, n) = 1\}.$$

The graph X_n is regular of degree $\varphi(n)$. Unitary Cayley graphs are highly symmetric and have some remarkable properties connecting graph theory and number theory. Fuchs [32] showed that the maximal length of an induced cycle in X_n is $2^k + 2$, where k is the number of different prime divisors of n . Klotz and Sander [25] determined the diameter, clique number, chromatic number and eigenvalues of unitary Cayley graphs.

In next four subsections, we will calculate the distance spectra and distance energy of X_n .

4.1 n is a prime number

In this case, X_n is a complete graph K_n with diameter 1. The spectra of X_n is $\{p-1, -1, -1, \dots, -1\}$ and

$$DE(X_n) = 2(n-1).$$

4.2 n is the power of 2

If $n = 2^k$ for $k > 1$, then X_n is a complete bipartite graph with the vertex partition

$$V(X_n) = \{0, 2, \dots, n-2\} \cup \{1, 3, \dots, n-1\},$$

and has diameter 2. The adjacency spectrum of X_n is $\{\frac{n}{2}, -\frac{n}{2}, 0, 0, \dots, 0\}$, and consequently the distance spectrum of X_n is $\{\frac{3n}{2} - 2, \frac{n}{2} - 2, -2, -2, \dots, -2\}$. Since $\varphi(2^k) = 2^{k-1}$, it follows that

$$DE(X_n) = \left| \frac{3n}{2} - 2 \right| + \left| \frac{n}{2} - 2 \right| + 2(n-2) = 4(n-2).$$

4.3 n is odd composite number

We need the following result from [25],

Theorem 4.1 *The number of common neighbors of distinct vertices a and b in the unitary Cayley graph X_n is given by $F_n(a - b)$, where $F_n(s)$ is defined as*

$$F_n(s) = n \prod_{i=1}^k \left(1 - \frac{\varepsilon(p)}{p} \right), \quad \text{with} \quad \varepsilon(p) = \begin{cases} 1 & \text{if } p \mid s \\ 2 & \text{if } p \nmid s \end{cases}$$

Since 2 does not divide n , according to Theorem 4.1 all factors in the expansion of $F_n(a - b)$ are greater than zero. Therefore, in this case there is a common neighbor of every pair of distinct vertices, which implies that the diameter is 2 and

$$D_n(X_n) = 2(J_n - I_n) - A_n(X_n),$$

where J_n is the matrix of ones and I_n is the identity matrix. The eigenvalues of D_n are given by

$$\mu_r = 2 \sum_{i=0}^{n-1} \omega_n^{ir} - 2 - c(r, n).$$

Finally, using the relation (2) the spectra of D_n is

$$\{2(n-1) - \varphi(n), -2 - c(1, n), -2 - c(2, n), \dots, -2 - c(n-1, n)\}$$

Therefore, the distance spectral radius of X_n is $2(n-1) - \varphi(n)$. We can calculate the distance energy of X_n using similar technique as in [33]. The nullity of a graph G , denoted as $\eta(G)$, is the multiplicity of zero as the eigenvalue in adjacency spectra.

Lemma 4.2 *The nullity of X_n is $n - m$, where $m = p_1 p_2 \cdots p_k$ is the maximal square-free divisor of n .*

Consider the following sum

$$S = \sum_{i=1}^n | -c(i, n) - 2 | = \sum_{i=1}^n |c(i, n) + 2|.$$

We already know the nullity of X_n , and therefore we will sum only the non-zero eigenvalues from the spectra of X_n . Divide the sum S in two parts: when $\frac{n}{\gcd(n, i)}$ is a square-free number with an even number of divisors and when $\frac{n}{\gcd(n, i)}$ is a square-free number with an odd number of divisors. The number of even subsets of $\{p_1, p_2, \dots, p_k\}$ is equal to the number of odd subsets of $\{p_1, p_2, \dots, p_k\}$.

In the first case, we have

$$\sum_{i \in S_1} \left(\frac{\varphi(n)}{\varphi\left(\frac{n}{\gcd(i,n)}\right)} + 2 \right) = \varphi(n) \cdot 2^{k-1} + 2 \sum_{l|m, \mu(l)=1} \varphi(l). \quad (3)$$

Let l be a square-free number that divides m with an even number of prime factors. The number of solutions of the equation $\frac{n}{\gcd(i,n)} = l$ is equal to $\varphi(l)$. For all $0 \leq i < n$ that satisfy $n = l \cdot \gcd(i, n)$ we have

$$\frac{\varphi(n)}{\varphi\left(\frac{n}{\gcd(i,n)}\right)} \cdot \varphi(l) + 2\varphi(l) = \varphi(n) + 2\varphi(l).$$

After taking the summation for all $l | m$ with $\mu(l) = 1$ we derive the identity (3). Analogously, in the second case we have

$$\sum_{i \in S_2} \left(\frac{\varphi(n)}{\varphi\left(\frac{n}{\gcd(i,n)}\right)} - 2 \right) = \varphi(n) \cdot 2^{k-1} - 2 \sum_{l|m, \mu(l)=-1} \varphi(l).$$

Since Euler function $\varphi(n)$ is multiplicative, after adding the above sums and eigenvalue -2 we get

$$\begin{aligned} S &= 2 \cdot (n - m) + \varphi(n) \cdot 2^k + 2 \sum_{l|m} \mu(l) \varphi(l) \\ &= 2n - 2m + \varphi(n) \cdot 2^k + 2 \prod_{i=1}^k (1 - \varphi(p_i)) = 2n - 2m + \varphi(n) \cdot 2^k + 2 \prod_{i=1}^k (2 - p_i). \end{aligned}$$

Finally, the distance energy of X_n equals

$$\begin{aligned} DE(X_n) &= S - |\varphi(n) + 2| + |2n - \varphi(n) - 2| \\ &= 2 \left(2n + \varphi(n)(2^{k-1} - 1) - m - 2 + \prod_{i=1}^k (2 - p_i) \right). \end{aligned}$$

4.4 n is even with odd prime divisor

As in the previous cases there are $\varphi(n)$ ones in the first row of matrix D_n . The vertex 0 is not adjacent to even vertices $2, 4, \dots, n-2$, but any two even vertices have a common neighbor by Theorem 4.1. Since all neighbors of an odd vertex are even vertices, we conclude that the number of vertices at distance 2 is exactly $\frac{n}{2} - 1$. Similarly, any two vertices a and b of X_n , which are both odd have a common neighbor. This proves that the diameter of X_n is 3 and there are exactly $\frac{n}{2} - \varphi(n)$ vertices at distance 3 from the starting vertex 0.

It follows that the polynomial $P_n(z)$ equals

$$P_n(z) = 3 \sum_{i=0}^{n-1} z^i - \sum_{i=0}^{\frac{n}{2}-1} z^{2i} - 2 - 2 \sum_{\substack{i=1 \\ \gcd(i,n)=1}}^n z^i.$$

The eigenvalues of D_n are given by

$$\mu_r = 3 \sum_{i=0}^{n-1} \omega_n^{ir} - \sum_{i=0}^{\frac{n}{2}-1} \omega_n^{2ir} - 2 - 2 \cdot c(r, n).$$

For $r = 0$, the distance spectral radius is equal to

$$\mu_0 = 3n - \frac{n}{2} - 2 - 2\varphi(n) = \frac{5n}{2} - 2(\varphi(n) + 1).$$

For $r = \frac{n}{2}$, it follows that $c(n/2, n) = -\varphi(n)$ and

$$\mu_{n/2} = 0 - \frac{n}{2} - 2 - 2 \cdot c\left(\frac{n}{2}, n\right) = 2(\varphi(n) - 1) - \frac{n}{2}.$$

For $r \neq 0$ and $r \neq \frac{n}{2}$, the value of the sum

$$\sum_{i=0}^{\frac{n}{2}-1} \omega_n^{2ir} = \frac{\omega_n^{nr} - 1}{\omega_n^{2r} - 1}$$

is zero. Finally, the spectra of D_n consists of $\frac{5n}{2} - 2(\varphi(n) + 1)$, $2(\varphi(n) - 1) - \frac{n}{2}$ and $-2 - 2 \cdot c(1, n), -2 - 2 \cdot c(2, n), \dots, -2 - 2 \cdot c(n/2 - 1, n), -2 - 2 \cdot c(n/2 + 1, n), \dots, -2 - 2 \cdot c(n - 1, n)$.

The sum $S' = \sum_{i=0}^{n-1} |c(i, n) + 1|$ can be computed analogously, and since the smallest prime of n is $p_1 = 2$ we have

$$S' = n - m + \varphi(n) \cdot 2^k + \prod_{i=1}^k (2 - p_i) = n - m + \varphi(n) \cdot 2^k.$$

The distance energy of X_n is equal to

$$\begin{aligned} DE(X_n) &= 2S' - |2 + 2 \cdot c(0, n)| - |2 + 2 \cdot c(n/2, n)| + \left| \frac{5n}{2} - 2(\varphi(n) + 1) \right| + \left| 2(\varphi(n) - 1) - \frac{n}{2} \right| \\ &= 2n - 2m + \varphi(n) \cdot 2^{k+1} - (2 + 2\varphi(n)) - (2\varphi(n) - 2) + \left(\frac{5n}{2} - 2(\varphi(n) + 1) \right) \\ &\quad + \left| 2(\varphi(n) - 1) - \frac{n}{2} \right|. \end{aligned}$$

The value of $|2(\varphi(n) - 1) - \frac{n}{2}|$ cannot be resolved, since it takes both positive, zero and negative values (consider for example $n = 6$, $n = 10$ and $n = 12$). By using the representation $n = 2^k \cdot m$ with m being odd, it follows $\varphi(n) = \varphi(2^k)\varphi(m) = 2^{k-1}\varphi(m)$. Therefore, the equation $2(\varphi(n) - 1) = \frac{n}{2}$ is equivalent with

$$2\varphi(n) = n + 1. \tag{4}$$

The equation (4) is related to the still open conjecture of Lehmer [34], and some obvious solutions involve prime Fermat numbers, $n = F_k - 2 = 2^{2^k} - 1$ for $k = 0, 1, 2, 3, 4, 5$.

5 Distance equienergetic graphs

Let $n = pq$, for odd prime numbers p and q . According to Subsection 4.3, the distance energy of $ICG_n(1)$ equals

$$\begin{aligned} DE(ICG_n(1)) &= 3pq + 2(p-1)(q-1) - n + 2(2-p)(2-q) \\ &= 6(p-1)(q-1). \end{aligned}$$

Consider the graph $ICG_n(1, p)$ – its diameter is two, and only the vertices satisfying $\gcd(a - b, n) = q$ are at distance 2. Using the formula (2) we get the distance eigenvalues of $ICG_n(1, p)$,

$$\mu_r = c(r, pq) + c(r, q) + 2 \cdot c(r, p).$$

Based on the greatest common divisor of r and pq , it follows that

$$\mu_r = \begin{cases} 1 - 1 + 2 \cdot (-1) & \text{if } p \nmid r \text{ and } q \nmid r \\ -\varphi(p) - 1 + 2 \cdot \varphi(p) & \text{if } p \mid r \text{ and } q \nmid r \\ -\varphi(q) + \varphi(q) + 2 \cdot (-1) & \text{if } p \nmid r \text{ and } q \mid r \\ \varphi(pq) + \varphi(q) + 2 \cdot \varphi(p) & \text{if } p \mid r \text{ and } q \mid r \end{cases}$$

Finally, the spectra of $ICG_n(1, p)$ consists of -2 with multiplicity $pq - q$, $p - 2$ with multiplicity $q - 1$, and $pq + p - 2$ with multiplicity 1. Therefore, the distance energy equals

$$\begin{aligned} DE(ICG_n(1, p)) &= q(p - 1) \cdot 2 + (q - 1) \cdot (p - 2) + 1 \cdot (pq + p - 2) \\ &= 4q(p - 1). \end{aligned}$$

Only for $q = 3$ and $n = 3p$, we have the identity $DE(ICG_n(1)) = DE(ICG_n(1, p))$, which represents a family of distance equienergetic integral circulant graphs. Note that $ICG_{3p}(1)$ is a subgraph of $ICG_{3p}(1, p)$ for all primes $p > 3$.

Let now $n = 2pq$, where p and q are different odd prime numbers. We will prove that graphs $ICG_n(1, p)$ and $ICG_n(1, q)$ have equal distance energy,

$$DE(ICG_n(1, p)) = DE(ICG_n(1, q)) = 4(3pq - p - q - 1).$$

According to Subsection 4.4, the distance eigenvalues of bipartite $ICG_n(1, p)$ are given by

$$\mu_r = c(r, 2pq) + c(r, 2q) + 2c(r, pq) + 2c(r, p) + 2c(r, q) + 3c(r, 2p) + 3c(r, 2).$$

Based on the greatest common divisor of r and $2pq$, it follows that

$$\mu_r = \begin{cases} -1 + 1 + 2 - 2 - 2 + 3 - 3 & \text{if } 2 \nmid r \text{ and } p \nmid r \text{ and } q \nmid r \\ 1 - 1 + 2 - 2 - 2 - 3 + 3 & \text{if } 2 \mid r \text{ and } p \nmid r \text{ and } q \nmid r \\ (p - 1) + 1 - 2(p - 1) + 2(p - 1) - 2 - 3(p - 1) - 3 & \text{if } 2 \nmid r \text{ and } p \mid r \text{ and } q \nmid r \\ -(p - 1) - 1 - 2(p - 1) + 2(p - 1) - 2 + 3(p - 1) + 3 & \text{if } 2 \mid r \text{ and } p \mid r \text{ and } q \nmid r \\ (q - 1) - (q - 1) - 2(q - 1) - 2 + 2(q - 1) + 3 - 3 & \text{if } 2 \nmid r \text{ and } p \nmid r \text{ and } q \mid r \\ -(q - 1) + (q - 1) - 2(q - 1) - 2 + 2(q - 1) - 3 + 3 & \text{if } 2 \mid r \text{ and } p \nmid r \text{ and } q \mid r \\ pq - 2p - 2 & \text{if } 2 \nmid r \text{ and } p \mid r \text{ and } q \mid r \\ 3pq + 2p - 2 & \text{if } 2 \mid r \text{ and } p \mid r \text{ and } q \mid r \end{cases}$$

$$\mu_r = \begin{cases} 3pq + 2p - 2 & \text{for } r = 0 \\ pq - 2p - 2 & \text{for } r = pq \\ -2p - 2 & \text{for } \gcd(r, 2q) = 1 \\ 2p - 2 & \text{for } \gcd(r, q) = 1 \\ -2 & \text{otherwise} \end{cases}$$

Since the multiplicity of distance eigenvalues $2p - 2$ and $-2p - 2$ is $q - 1$, we finally get

$$\begin{aligned} DE(ICG_n(1, p)) &= (3pq + 2p - 2) + (pq - 2p - 2) + (q - 1)(2p + 2) + (q - 1)(2p - 2) + 2(pq - 2q) \\ &= 12pq - 4p - 4q - 4, \end{aligned}$$

which is a symmetric expression involving p and q . Note that the diameter of graphs $ICG_{2pq}(1, p)$ and $ICG_{2pq}(1, q)$ is equal to three.

These constructions of infinite families of distance equienergetic graphs are the first ones derived not using the product of graphs nor iterated line graphs. It would be of great interest to calculate the distance energy of other classes of integral circulant graphs, and explore new families of non-cospectral distance equienergetic integral graphs.

Acknowledgement. This work was supported by the research grant 144007 of the Serbian Ministry of Science and Technological Development. The author is grateful to the anonymous referees for their comments and suggestions.

References

- [1] I. Gutman, *The energy of a graph*, Ber. Math. Stat. Sekt. Forschungszent. Graz 103 (1978), 1–22.
- [2] I. Gutman, *The energy of a graph: old and new results*, Algebraic Combinatorics and Applications, Springer Berlin, 2001, 196–211.
- [3] R. A. Brualdi, *Energy of a Graph*, available from <http://www.public.iastate.edu/~lhogben/energyB.pdf>
- [4] F. Buckley, F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood, 1990.
- [5] R. L. Graham, H. O. Pollack, *On the addressing problem for loop switching*, Bell System Tech. J. 50 (1971), 2495–2519.
- [6] R. L. Graham, L. Lovász, *Distance matrix polynomials of trees*, Adv. in Math. 29 (1978), 60–88.
- [7] R. B. Bapat, *Distance matrix and Laplacian of a tree with attached graphs*, Linear Algebra Appl. 411 (2005), 295–308.
- [8] R. B. Bapat, S. J. Kirkland, M. Neumann, *On distance matrices and Laplacians*, Linear Algebra Appl. 401 (2005), 193–209.
- [9] B. Zhou, *On the largest eigenvalue of the distance matrix of a tree*, MATCH Commun. Math. Comput. Chem. 58 (2007), 657–662.
- [10] G. Indulal, *Sharp bounds on the distance spectral radius and the distance energy of graphs*, Linear Algebra Appl. 430 (2009), 106–113.
- [11] D. Stevanović, A. Ilić, *Distance spectral radius of trees with fixed maximum degree*, Electron. J. Linear Algebra 20 (2010), 168–179.
- [12] V. Consonni, R. Todeschini, *New spectral indices for molecule description*, MATCH Commun. Math. Comput. Chem. 60 (2008), 3–14.
- [13] V. Brankov, D. Stevanović, I. Gutman, *Equienergetic chemical trees*, J. Serb. Chem. Soc. 69 (2004), 549–553.
- [14] H. S. Ramane, H. B. Walikar, *Construction of equienergetic graphs*, MATCH Commun. Math. Comput. Chem. 57 (2007), 203–210.
- [15] L. Xu, Y. Hou, *Equienergetic bipartite graphs*, MATCH Commun. Math. Comput. Chem. 57 (2007), 363–370.
- [16] A. S. Bonifácio, C. T. M. Vinagre, N. M. M. de Abreu, *Constructing pairs of equienergetic and non-cospectral graphs*, Appl. Math. Letters 21 (2008), 338–341.
- [17] G. Indulal, I. Gutman, A. Vijayakumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem. 60 (2008), 461–472.
- [18] H. S. Ramane, D. S. Revankar, I. Gutman, H. B. Walikar, *Distance spectra and distance energies of iterated line graphs of regular graphs*, Publ. Inst. Math. 85 (2009), 39–46.
- [19] H. S. Ramane, I. Gutman, D. S. Revankar, *Distance equienergetic graphs*, MATCH Commun. Math. Comput. Chem. 60 (2008), 473–484.

- [20] D. Stevanović, G. Indulal, *The distance spectrum and energy of the compositions of regular graphs*, Appl. Math. Letters 22 (2009), 1136–1140.
- [21] K. Balińska, D. Cvetković, Z. Radosavljević, S. Simić, D. Stevanović, *A survey on integral graphs*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 13 (2002), 42–65.
- [22] N. Saxena, S. Severini, I. Shparlinski, *Parameters of integral circulant graphs and periodic quantum dynamics*, Int. J. Quant. Inf. 5 (2007), 417–430.
- [23] C. D. Godsil, *Periodic Graphs*, arXiv:0806.2074v1 [math.CO], 12 Jun 2008.
- [24] W. So, *Integral circulant graphs*, Discrete Math. 306 (2006), 153–158.
- [25] W. Klotz, T. Sander, *Some properties of unitary Cayley graphs*, Electron. J. Combin. 14 (2007), #R45
- [26] M. Bašić, M. Petković, D. Stevanović, *Perfect state transfer in integral circulant graphs*, Appl. Math. Letters 22 (2009), 1117–1121.
- [27] M. Bašić, M. D. Petković, *Some classes of integral circulant graphs either allowing or not allowing perfect state transfer*, Appl. Math. Letters 22 (2009), 1609–1615.
- [28] M. Bašić, A. Ilić, *On the clique number of integral circulant graphs*, Appl. Math. Letters, 22 (2009), 1406–1411.
- [29] P. J. Davis, *Circulant matrices*, Pure and Applied Mathematics, John Wiley & Sons, New York-Chichester-Brisbane, 1979.
- [30] G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers*, Fifth edition, Oxford University Press, New York, 1980.
- [31] A. Dobrynin, R. Entringer, I. Gutman, *Wiener index of trees: theory and applications*, Acta Appl. Math. 66 (2001), 211–249.
- [32] E. D. Fuchs, *Longest induced cycles in circulant graphs*, Electron. J. Combin. 12 (2005), 1–12.
- [33] A. Ilić, *The energy of unitary Cayley graphs*, Linear Algebra Appl. 431 (2009), 1881–1889.
- [34] D. H. Lehmer, *On Euler's totient function*, Bull. Amer. Math. Soc. 38 (1932), 745–751.